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THE ASYMPTOTIC RELATIVE EFFICIENCY OF VARIOUS NONPARAMETRIC DETECTORS USED ON SPECTRAL DATA

M.J. Wilmut and R.F. MacKinnon

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10 M. J. Wilmut [redacted] R. F. MacKinnon

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ABSTRACT

Asymptotic relative efficiency (ARE) and asymptotic loss are two ways to compare the performance of one detector to another and in particular to the optimum detector.

Various authors have suggested and studied a variety of nonparametric detection schemes applicable in the analysis of spectral data. Here we derive the ARE's of the modified Savage detector and Sc set of detectors. It is shown that in most cases of practical interest the ARE of the modified Savage detector is higher than that of the Mann-Whitney detector. The asymptotic loss of the modified Savage detector with respect to the optimum detector in these cases is less than 1 dB.

INTRODUCTION

We consider the situation (References (1) to (4)) in which a signal embedded in a time series appears as a narrow line in the frequency domain. Detection is based on the matrix of the spectral estimates X_{ij} : $i=1,2,\ldots M$; $j=1,2,\ldots,N$. Here time is referenced by the i parameter and frequency by the j parameter. The parameter N is the largest number of frequency estimates for which the spectrum can be said to be flat, while M is the number of spectral estimates per frequency cell to be used in the decision process.

Let R_{ij} be the rank of the X_{ij} for a fixed time interval; that is i fixed, $j=1,2,\ldots,N$. The null hypothesis is that all N frequencies contain noise only. For the X_{ij} independent identically distributed random variables, the R_{ij} are uniformly distributed over the integers $1,2,\ldots N$. The alternative hypothesis is that a signal occurs at one of the frequencies represented. In order to determine the distribution of the ranks in this case, one must assume a specific distribution for the X_{ij}; that is, one must specify the nature of the signal and the noise.

Let $P(k)=P(k,N,\theta)$ be the probability that the signal frequency takes rank k where $k=1,2,\ldots N$. The parameter θ is the signal-to-noise ratio. In (2) and (4) it is shown that:

$$P(k)=P(k,N,\alpha)=\frac{\alpha\Gamma(N)\,\Gamma(N+\alpha-k)}{\Gamma(N+\alpha)\Gamma(N-k+1)} \tag{1a}$$

where α represents $\frac{1}{1+\theta}$ when the signal is narrow-band with a gaussian amplitude distribution (also called a Rayleigh Fading signal).

$$P(k)=P(k,N,\theta)=\frac{\Gamma(N)\exp(-\theta)}{\Gamma(N-k+1)}\sum_{i=0}^{k-1}\frac{(-1)^i\exp(\theta/(N+1-k+i))}{(N+1-k+i)\Gamma(i+1)\Gamma(k-i)} \tag{1b}$$

when the signal is a sinusoid (also called a no fading signal).

$$P(k)=P(k,N,a,b)=\frac{b^2\,\Gamma(N-k+b)\,\Gamma(N)}{\Gamma(N+b)\,\Gamma(N-k+1)}\left[1+2a\sum_{i=0}^{k-1}(i+N+b-k)^{-1}\right] \tag{1c}$$

where a is equal to $(2+4/\theta)^{-1}$ and b is equal to $(1+\frac{1}{2}\theta)^{-1}$. This probability function results when we have a narrowband signal with a fluctuating amplitude (also referred to as a Swerling case 4 signal). The standard deviation divided by the mean is a measure of fluctuation of a signal. This quantity is 1, 0 and $(2)^{-\frac{1}{2}}$ respectively for the above signal spectral estimates.

<u>Detector</u>

The modified nonparametric decision schemes discussed below are based on the sum of M independent random variables representing some function of the rank at each frequency. That is, we decide that a signal is present at frequency j depending on the value of

$$W_j = \sum_{i=1}^{M} a(R_{ij})$$

where a (x) is some function of x.

Suggested forms for a (x) include:

(i) $a(R_{ij})=R_{ij}$ (modified Mann-Whitney statistic).

(ii) $a(R_{ij})=\sum_{\ell=N-R_{ij}+1}^{N} \ell^{-1}$ (modified Savage test).

(iii) $a(R_{ij})=\mathrm{sign}(R_{ij}-c)$ (Sc detector). Here c is any integer from 1 to N. In other words we set the largest c spectral estimates to one; all other estimates are given the value zero.

Asymptotic Relative Efficiency

The efficiency of a detector is defined (5) as

$$e = \lim_{M\to\infty} \frac{1}{M} \frac{\left[\frac{\partial}{\partial\theta} E(W|\theta)\Big|_{\theta=0}\right]^2}{\sigma_o^2(W)} \qquad (2)$$

where $\sigma_o^2(W)$ is the variance of the test statistic W in the noise only case and $E(W|\theta)$ the expected value of W when the signal-to-noise ratio is θ.

To compare two detectors A and B, we form the ratio of their efficiencies $E_{B,A}={}^{e}B/e_A$ and this is then the asymptotic relative efficiency of detector B with respect to detector A. The ARE is a measure of the performance of one detector with respect to another when a large number of observations are available for processing and the signal-to-noise ratio is small. Even when a small number of observations are available the ARE in many situations of practical interest is a good measure of the relative efficiencies of the detectors being studied (6).

The asymptotic loss of detector A compared to detector B is given by (2)

$$L_\infty = \lim_{M\to\infty} \left[\frac{S_A(Pd,Pf,M)}{S_B(Pd,Pf,M)}\right]$$

where $S_A(Pd,Pf,M)$ is the signal-to-noise ratio required by detector A to yield specified values of Pd(probability of detection) and Pf (false alarm probability) for a given number of observations M. In other words, it is the asymptotic degradation of signal-to-noise ratio when one uses one detector as compared to using another (when both must meet the same performance criteria and the number of observations available is large).

The ARE and asymptotic loss in our problem are related by $L_\infty = E^{-1/2}$ or in decibels $L_\infty = -5 \log_{10} E$.

ASYMPTOTIC RELATIVE EFFICIENCY WHEN SIGNAL GAUSSIAN

In the following we determine the quantities $\sigma_o^2(W)$ and $\frac{\partial}{\partial\theta}E(W|\theta)$ (ignoring the subscript j) in order to obtain the efficiencies (Equation(2)) and hence AREs of the test statistics noted above. We assume that the signal is gaussian.

Modified Savage Detector

Let S in this section denote the modified Savage random variable $a(R_{ij})= \sum_{\ell=N-R_{ij}+1}^{N} \ell^{-1}$. We will determine $\sigma_o^2(S)$ and $E(S|\theta)$ and as W is the sum of M such independent random variables $\sigma_o^2(W)=M\sigma_o^2(S)$ and $E(W|\theta)=ME(S|\theta)$. We now state and prove two propositions needed to determine the ARE of the Savage detector.

Proposition 1: The expected value of the statistic S for a given value of θ is given by

$$E(S|\theta)= \frac{1}{N} + \frac{1}{\alpha} - \frac{\Gamma(\alpha)\Gamma(N)}{\Gamma(N+\alpha)}$$

where $\alpha = \frac{1}{1+\theta}$.

To determine $E(S|\theta)$ we must evaluate $\sum_{k=1}^{N} \beta_{k,N} P(k,N,\alpha)$ where $\beta_{k,N}= \sum_{M=N-k+1}^{N} \frac{1}{M}$ is the weighting for the Savage statistic and $P(k,N,\alpha)$ is from Equation (1a). Let

$$T_N = \sum_{k=1}^{N} \beta_{k,N} P(k)= \sum_{k=1}^{N} \left(\sum_{M=N-k+1}^{N} \frac{1}{M} \right) \frac{\alpha\Gamma(N)\Gamma(N+\alpha-k)}{\Gamma(N+\alpha)\Gamma(N-k+1)}$$

$$=\alpha \frac{1}{N} \frac{\Gamma(N)\Gamma(N+\alpha-1)}{\Gamma(N+\alpha)\Gamma(N)} + \sum_{k=2}^{N} \left(\sum_{M=N-k+1}^{N} \frac{1}{M} \right) \frac{\alpha\Gamma(N)\Gamma(N+\alpha-k)}{\Gamma(N+\alpha)\Gamma(N-1+1)}$$

$$= \frac{\alpha}{N(N+\alpha-1)} + \sum_{k=2}^{N} \left(\sum_{M=N-k+1}^{N-1} \frac{1}{M} \right) \frac{\alpha\Gamma(N)\Gamma(N+\alpha-k)}{\Gamma(N+\alpha)\Gamma(N-k+1)} + \sum_{k=2}^{N} \frac{1}{N} \frac{\alpha\Gamma(N)\Gamma(N+\alpha-k)}{\Gamma(N+\alpha)\Gamma(N-k+1)}$$

$$= \frac{\alpha}{N(N+\alpha-1)} + \sum_{k=1}^{N-1} \left(\sum_{M=N-k}^{N-1} \frac{1}{M} \right) \frac{\alpha(N-1)\Gamma(N-1)\Gamma(N+\alpha-k-1)}{(N+\alpha-1)\Gamma(N+\alpha-1)\Gamma(N-k)} + \frac{\alpha}{N} \sum_{k=1}^{N-1} \frac{(N-1)\Gamma(N-1)\Gamma(N+\alpha-k-1)}{(N+\alpha-1)\Gamma(N+\alpha-1)\Gamma(N-k)}$$

$$= \frac{\alpha}{N(N+\alpha-1)} + \frac{N-1}{N+\alpha-1} T_{N-1} + \frac{(N-1)}{N(N+\alpha-1)} = \frac{1}{N} + \frac{(N-1)}{N+\alpha-1} T_{N-1}.$$

That is, T_N, the mean of the modified Savage statistic when N frequencies are ranked, satisfies the recursion relation

$$T_N = \frac{1}{N} + \frac{(N-1)}{N+\alpha-1} T_{N-1}.$$

It is easily verified that

$T_N = \frac{1}{N} + \frac{1}{\alpha} - \frac{\Gamma(\alpha)\Gamma(N)}{\Gamma(N+\alpha)}$ satisfies this relation, or that

$$E(S|\theta) = \frac{1}{N} + \frac{1}{\alpha} - \frac{\Gamma(\alpha)\Gamma(N)}{\Gamma(N+\alpha)} \text{ where } \alpha = \frac{1}{1+\theta}.$$

<u>Corollary 1</u>:

Since $E(S|\theta) = \frac{1}{N} + \frac{1}{\alpha} - \frac{\Gamma(\alpha)\Gamma(N)}{\Gamma(N+\alpha)}$ where $\alpha = \frac{1}{1+\theta}$ and $\Gamma'(Z) = \Psi(Z)\Gamma(Z)$ where $\Psi(Z)$ is the digamma function (7) it is easily verified that

$$\left.\frac{\partial E(S|\theta)}{\partial \theta}\right|_{\theta=0} = 1 - \frac{1}{N}(\Psi(N+1) - \Psi(1))$$

$$= 1 - \frac{1}{N} \sum_{k=1}^{N} \frac{1}{k}.$$

We note that $E(S|0) = 1$ or the mean of S under the null hypothesis is one.

<u>Proposition 2</u>: The variance of the statistic S under the null hypothesis is given by

$$\sigma_o^2(S) = 1 - \frac{1}{N} \sum_{J=1}^{N} \frac{1}{J}.$$

First we prove $E(S^2) = 2 - \frac{1}{N} \sum_{J=1}^{N} \frac{1}{J}$ (3)

and then as $E(S|0) = 1$, $\sigma_o^2(S)$ is easily found.

Now

$$E(S^2) = \sum_{k=1}^{N} \left(\sum_{J=N-k+1}^{N} \frac{1}{J} \right)^2 \frac{1}{N}. \tag{4}$$

We show $E(S^2)$ defined by (4) reduces to (3) by mathematical induction. By direct calculation (4) reduces to (3) when N is 2. Next we assume the result holds for N-1 and prove it then holds for N.

Since

$$\left(\sum_{J=N-k}^{N-1} \frac{1}{J} \right)^2 = \left(\sum_{J=N-k}^{N-1} \frac{1}{J} + \frac{1}{N} \right)^2 - \frac{1}{N^2} - \frac{2}{N} \sum_{J=N-k}^{N-1} \frac{1}{J}$$

for any k from 1 to N-1 we may write

$$\sum_{k=1}^{N-1} \left(\sum_{J=N-k}^{N-1} \frac{1}{J} \right)^2 = \sum_{k=1}^{N-1} \left(\sum_{J=N-k}^{N-1} \frac{1}{J} + \frac{1}{N} \right)^2 - \frac{N-1}{N^2} - \frac{2}{N} (N-1)$$

$$= \sum_{k=1}^{N} \left(\sum_{J=N-k+1}^{N} \frac{1}{J} \right)^2 - 2 + \frac{1}{N}.$$

That is,

$$\frac{1}{N} \sum_{k=1}^{N} \left(\sum_{J=N-k+1}^{N} \frac{1}{J} \right)^2 = \frac{1}{N} \sum_{k=1}^{N-1} \left(\sum_{J=N-k}^{N-1} \frac{1}{J} \right)^2 + \frac{2}{N} - \frac{1}{N^2}.$$

By assumption

$$\sum_{k=1}^{N-1} \left(\sum_{J=N-k}^{N-1} \frac{1}{J} \right)^2 = 2(N-1) - \sum_{J=1}^{N-1} \frac{1}{J}$$

and hence

$$\frac{1}{N} \sum_{k=1}^{N} \left(\sum_{J=N-k+1}^{N} \frac{1}{N} \right)^2 = \frac{1}{N} \left[2(N-1) - \sum_{J=1}^{N-1} \frac{1}{J} \right] + \frac{2}{N} - \frac{1}{N^2} = 2 - \frac{1}{N} \sum_{J=1}^{N} \frac{1}{J}.$$

The results of the above propositions now enable us to evaluate the efficiency of the Savage detector using Equation (2).

$$e_S = 1 - \frac{1}{N} \sum_{J=1}^{N} \frac{1}{J} \cdot$$

It can be shown (2) that the efficiency of the optimum detector is unity.

Thus the ARE of the Savage detector with respect to the optimum is given by

$$E_{S,opt} = 1 - \frac{1}{N} \sum_{J=1}^{N} \frac{1}{J} \cdot \qquad (5a)$$

<u>Sc Detector</u>

Let Sc in this section denote the Sc random variable $a(R_{ij}) = \text{sign } (R_{ij}-c)$. The mean of Sc for a given signal-to-noise ratio is then

$$E(Sc|\theta) = \sum_{k=N-c+1}^{N} P(k,N,\alpha \text{ where } P(k,N,\alpha) \text{ is given by Equation (1a).}$$

It then follows that

$$\left.\frac{\partial P(k,N,\theta)}{\partial\theta}\right|_{\theta=0} = \frac{1}{N} (\Psi(N+1) - \Psi(N+1-k) - 1) \cdot$$

Hence

$$\left.\frac{\partial E(Sc|\theta)}{\partial\theta}\right|_{\theta=0} = \frac{1}{N} \sum_{k=N-c+1}^{N} (\Psi(N+1) - \Psi(N+1-k) - 1)$$

$$= \frac{1}{N} (c\Psi(N+1) - \{c\Psi(c) - c + 1\} - c)$$

$$= \frac{c}{N} (\Psi(N+1) - \Psi(c+1)).$$

Under the null hypothesis $P(k,N,1) = \frac{1}{N}$ for all k.

Hence $E(Sc|0) = c/N$ and $E(Sc^2|0) = c/N$ as well. Thus the variance under the null hypothesis is given by

$$\sigma_o^2(Sc) = \frac{c}{N} (1 - c/N).$$

Substitution of the above quantities in Equation (2) gives the efficiency of this type of detector. Thus

$$e_{Sc} = \frac{1}{\frac{c}{N}\left(\frac{N-c}{N}\right)} \left(\frac{c}{N}(\Psi(N+1) - \Psi(c+1)\right)^2$$

$$= \frac{N}{N-c} (\Psi(N+1) - \Psi(c+1))^2$$

$$= \frac{c}{N-c} \left[\sum_{J=1}^{N-c} \frac{1}{c+J} \right]^2 .$$

Once again the efficiency of the optimum receiver is unity. Thus the ARE of the Sc detector is

$$E_{Sc,opt} = \frac{c}{N-c} \left[\sum_{J=1}^{N-c} \frac{1}{c+J} \right]^2 . \tag{5b}$$

The ARE of the detector when $c = 1$ has been determined by Hansen and Olsen (2) and agrees with this formula.

Table I below tabulates for various values of N the c which corresponds to the detector with maximum ARE. As N tends to infinity the c corresponding to the maximum ARE is given by $c \doteq .2032\, N$ with an ARE of .6476.

ASYMPTOTIC RELATIVE EFFICIENCY FOR FLUCTUATING AND SINUSOIDAL SIGNALS

It is easily verified that $\left.\frac{\partial P(k,N,\theta)}{\partial \theta}\right|_{\theta=0}$ is the same for all forms of $P(k,N,\theta)$ given in Equations (1a) (1b) and (1c) Thus $\left.\frac{\partial}{\partial \theta} E(W|\theta)\right|_{\theta=0}$ is the same for all signals. Also we have the identical null hypothesis for all types of signals. Hence the ARE's for the gaussian, sinusoidal and fluctuating signals are equal.

COMPARISON OF ASYMPTOTIC RELATIVE EFFICIENCIES

In (2) it is shown that the ARE of the Mann-Whitney detector (8) is given by $\frac{3}{4}\left[\frac{1}{1+1(N-1)^{-1}}\right]$ and that of the modified Mann-Whitney is

$\frac{3}{4}\left[\frac{1}{1+2(N-1)^{-1}}\right]$. Table I gives the ARE of these detectors, the modified Savage detector, the S1 detector and the Sc detector member with maximum ARE.

TABLE I. Asymptotic relative efficiency of various nonparametric detectors as compared to the optimum parametric detector.

N	Modified Savage	Mann-Whitney	Modified Mann-Whitney	Maximum of Sc and corresponding c		S1
2	.250	.375	.250	.250	(1)	.250
3	.388	.500	.375	.347	(1)	.347
4	.479	.563	.450	.391	(1)	.391
7	.630	.643	.563	.478	(2)	.423
8	.660	.656	.583	.494	(2)	.422
16	.789	.703	.662	.561	(4)	.378
32	.873	.727	.705	.602	(7)	.302
64	.926	.738	.727	.624	(14)	.223
∞	1	.75	.75	.6476	($c \doteq .2N$)	0

We note that the Mann-Whitney has the highest ARE if N is seven or less. However, for larger values of N the modified Savage is best. The number of estimates that must be ranked simultaneously to calculate the Mann-Whitney statistic, namely MN, makes this scheme unattractive in many practical problems. For all values of N the modified Savage ARE is as good as or better

than the modified Mann-Whitney ARE with equality only when N is two.

The most important result is that as N tends to infinity the ARE of the modified Savage statistic tends to one while the ARE of both the Mann-Whitney tests approaches 3/4.

Table II is a tabulation of the asymptotic loss in decibels (compared to the optimum parametric detector) for the modified Savage and Mann-Whitney statistics. We note that if N is eight or more this loss is less than 1 dB for the modified Savage detector.

TABLE II. Asymptotic loss of the modified Savage and Mann-Whitney detectors compared to the optimum parametric detector (in decibels).

N	Modified Savage	Mann-Whitney
2	3.0	3.0
4	1.60	1.25
8	.90	.92
16	.52	.77
32	.30	.69
64	.17	.66
∞	0	.63

CONCLUSIONS

The asymptotic performance for the modified Savage detector and Sc class of detectors has been obtained. Comparison by way of ARE of these detectors with other detectors commonly used is given in Table I.

The ARE of the Savage detector approaches 1 as N, the number of spectral estimates ranked, increases. We note that if N is seven or more the ARE of the modified Savage detector is higher than that of the Mann-Whitney, "usually considered to be one of the more effective non-parametric tests for use in two-sample detection situations" (2).

Table II shows that for values of N that might be used in a practical implementation (say N = 8, 16, 32, 64) the asymptotic loss of the modified Savage detector with respect to the optimum parametric detector is less than 1 dB (.90, .52, .30, .17 decibels respectively).

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